

Probabilistic properties of the elliptic motion

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Abstract

In this paper we consider the plane elliptic motion which occurs if the moving centrode is a circle of radius r and the fixed centrode a circle of radius $2r$. Every point of the moving plane generates an ellipse in the fixed plane. Let a disk of radius R , $0 \leq R < \infty$, concentric to the moving centrode be attached to the moving plane. If a point P is chosen at random from this disk, then the area and the perimeter of the ellipse generated by P are random variables. We determine the moments and the distributions of these random variables for the case that P is uniformly distributed over the area of the disk.

2010 Mathematics Subject Classification: 52A22, 53C65, 60D05, 53A17, 51N20, 33E05

Keywords: elliptic motion, random ellipses, area moments, area distribution, perimeter moments, perimeter distribution, centrode, elliptic integrals

1 Introduction

We consider a fixed Euclidean plane S with a Cartesian frame of origin O and x, y axes, and a moving Euclidean plane Σ with a Cartesian frame of origin Ω and ξ, η axes. To Σ we attach a circle C_1 with centre Ω and radius r ; to S we attach a circle C_0 with centre O and radius $2r$ (see Fig. 1).

If C_1 rolls inside C_0 , then every point P fixed in Σ generates an ellipse in S . Therefore, this motion is called *elliptic motion*. If $P = \Omega$, then the ellipse is a circle with centre O and radius r ; if $P \in C_1$, then the ellipse degenerates to a (double) line segment of length $4r$ which is one diameter of C_0 . [4, pp. 2-3, 8-9], [1, pp. 14-15]

We denote by φ the angle between the x -axis and the line segment $\overline{O\Omega}$. W.l.o.g. we assume that for $\varphi = 0$ the ξ -axis lies on the x -axis and both axes have equal direction. Then the equation in complex form of the ellipse generated by P is given by

$$X = X(\varphi) = r e^{i\varphi} + \Xi e^{-i\varphi}, \quad 0 \leq \varphi < 2\pi, \quad (1.1)$$

with

- 1) $\Xi := \varrho e^{i\alpha}$, where ϱ, α are the polar coordinates of P with respect to the ξ, η -frame, or
- 2) $\Xi := \xi + i\eta$, where ξ, η are the Cartesian coordinates of P with respect to the ξ, η -frame.

In the first case, from (1.1) we get

$$\left. \begin{aligned} x &= r \cos \varphi + \varrho \cos(\varphi - \alpha), \\ y &= r \sin \varphi - \varrho \sin(\varphi - \alpha) \end{aligned} \right\} \quad (1.2)$$

as parametric representation of the ellipse, and in the second case,

$$\left. \begin{aligned} x &= r \cos \varphi + \xi \cos \varphi + \eta \sin \varphi, \\ y &= r \sin \varphi + \eta \cos \varphi - \xi \sin \varphi. \end{aligned} \right\} \quad (1.3)$$

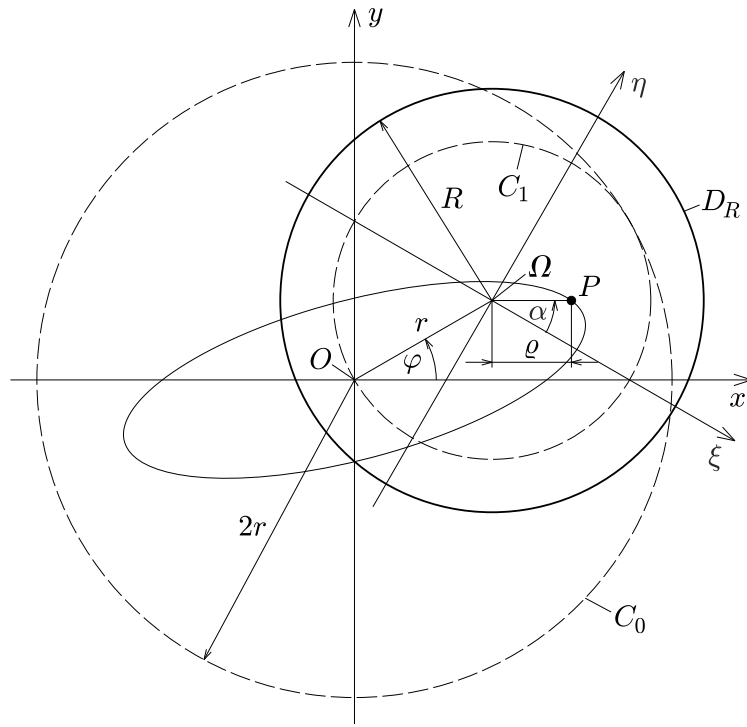


Fig. 1: Elliptic motion

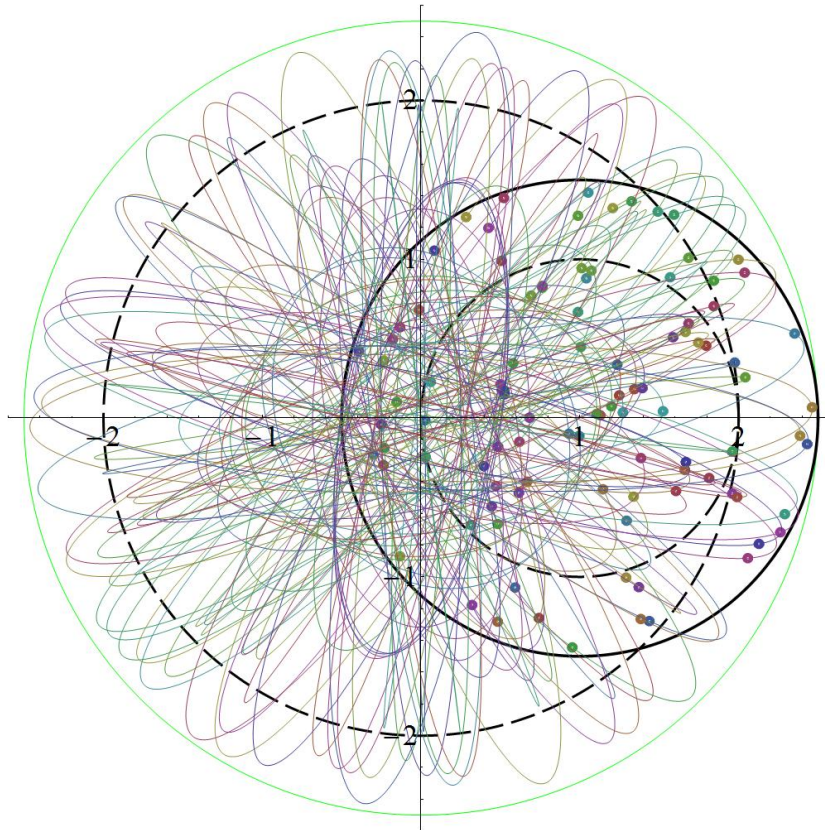


Fig. 2: 100 random ellipses and their generating points; $r = 1$, $R = 1.5$

From (1.2) one finds that the length of the semi-major axis is equal to $r + \varrho$, and the length of the semi-minor axis equal to $|r - \varrho|$. Hence all points $P \in \Sigma$ with equal distance ϱ from Ω generate congruent ellipses. The centres of all ellipses lie in O . The angle between the x -axis and the major-axis of an ellipse is equal to $\alpha/2$.

C_0 is the fixed centre and C_1 the moving centre of the elliptic motion. It is possible to get the equations of C_0 and C_1 backwards from the equation(s) of the motion (1.1), (1.2) or (1.3). [4, p. 8-9], [1, p. 14-15] (For the notions of the fixed and the moving centre see e.g [2, pp. 257-259].)

Now we consider a disk

$$D_R = \{(\xi, \eta) \in \Sigma : 0 \leq \xi^2 + \eta^2 \leq R^2\}, \quad 0 \leq R < \infty. \quad (1.4)$$

If P is chosen at random from D_R , then the area and the perimeter of the ellipse generated by P are random variables which we denote by A_R and U_R , respectively. In this paper we determine the moments and the distributions of these random variables for the case that P is uniformly distributed over the area of D_R . Since r is no essential parameter, we identify $A_\varkappa \equiv A_R$ and $U_\varkappa \equiv U_R$, where $\varkappa = R/r$

Fig. 2 shows the result of a simulation with 100 random points and ellipses.

2 Moments of the area

The moments of the random area A_R enclosed by the ellipse generated by a random point $P \in D_R$ (see (1.4)) are given by

$$\mathbb{E}[A_R^k] = \left(\int_{P \in D_R} A^k dP \right) / \left(\int_{P \in D_R} dP \right),$$

where A is the area of the ellipse generated by the point $P \in D_R$, and $dP = d\xi \wedge d\eta$ is the density for sets of points in the plane. Up to a constant factor this density is the only one that is invariant under motions [5, p. 13]. Due to the point symmetry we use the polar coordinates ϱ, α . With $\xi = \varrho \cos \alpha$, $\eta = \varrho \sin \alpha$ we get

$$\begin{aligned} d\xi &= \frac{\partial \xi}{\partial \varrho} d\varrho + \frac{\partial \xi}{\partial \alpha} d\alpha = \cos \alpha d\varrho - \varrho \sin \alpha d\alpha, \\ d\eta &= \frac{\partial \eta}{\partial \varrho} d\varrho + \frac{\partial \eta}{\partial \alpha} d\alpha = \sin \alpha d\varrho + \varrho \cos \alpha d\alpha \end{aligned}$$

and

$$\begin{aligned} d\xi \wedge d\eta &= \cos \alpha \sin \alpha d\varrho \wedge d\varrho + \varrho \cos^2 \alpha d\varrho \wedge d\alpha - \varrho \sin^2 \alpha d\alpha \wedge d\varrho - \varrho^2 \sin \alpha \cos \alpha d\alpha \wedge d\alpha \\ &= \varrho d\varrho \wedge d\alpha, \end{aligned}$$

hence

$$\mathbb{E}[A_R^k] = \frac{1}{\pi R^2} \int_{\alpha=0}^{2\pi} \int_{\varrho=0}^R A^k(\varrho) \varrho d\varrho d\alpha = \frac{2}{R^2} \int_{\varrho=0}^R A^k(\varrho) \varrho d\varrho. \quad (2.1)$$

The area enclosed by an ellipse generated by a point $P \in D_R$ with distance ϱ from Ω is given by

$$A(\varrho) = \begin{cases} \pi (r^2 - \varrho^2) & \text{if } 0 \leq \varrho \leq r, \\ \pi (\varrho^2 - r^2) & \text{if } r < \varrho < \infty. \end{cases} \quad (2.2)$$

With $w = \varrho/r$, the area (2.2) function may be written as

$$\tilde{A}(w) = \begin{cases} \pi r^2 (1 - w^2) & \text{if } 0 \leq w \leq 1, \\ \pi r^2 (w^2 - 1) & \text{if } 1 < w < \infty. \end{cases} \quad (2.3)$$

Theorem 2.1. *The k -th moment, $k = 1, 2, \dots$, of the random area $A_{\varkappa} \equiv A_R$, $\varkappa = R/r$, of an ellipse generated by a random point $P \in D_R$ (P uniformly distributed over the area of the disk D_R) is given by*

$$\mathbb{E}[A_{\varkappa}^k] = \begin{cases} \pi^k r^{2k} & \text{if } \varkappa = 0, \\ \frac{\pi^k r^{2k}}{k+1} \frac{1 - (1 - \varkappa^2)^{k+1}}{\varkappa^2} & \text{if } 0 < \varkappa \leq 1, \\ \frac{\pi^k r^{2k}}{k+1} \frac{1 + (\varkappa^2 - 1)^{k+1}}{\varkappa^2} & \text{if } 1 < \varkappa < \infty. \end{cases}$$

Proof. First, we consider the case $0 < \varkappa \leq 1$ ($0 < R \leq r$). Eq. (2.1) becomes

$$\mathbb{E}[A_R^k] = \frac{2\pi^k}{R^2} \int_0^R (r^2 - \varrho^2)^k \varrho \, d\varrho.$$

The substitution $\varrho = rw$ gives

$$\mathbb{E}[A_R^k] = \frac{2\pi^k r^{2k+2}}{R^2} \int_{w=0}^{R/r} (1 - w^2)^k w \, dw,$$

which, with $A_{\varkappa} \equiv A_R$ may also be written as

$$\mathbb{E}[A_{\varkappa}^k] = \frac{2\pi^k r^{2k}}{\varkappa^2} \int_{w=0}^{\varkappa} (1 - w^2)^k w \, dw.$$

The substitution

$$y = 1 - w^2, \quad dy = -2w \, dw, \quad dw = -\frac{dy}{2w}$$

yields

$$\begin{aligned} \mathbb{E}[A_{\varkappa}^k] &= -\frac{\pi^k r^{2k}}{\varkappa^2} \int_{y=1}^{1-\varkappa^2} y^k \, dy = \frac{\pi^k r^{2k}}{\varkappa^2} \int_{1-\varkappa^2}^1 y^k \, dy = \frac{\pi^k r^{2k}}{\varkappa^2(k+1)} y^{k+1} \Big|_{1-\varkappa^2}^1 \\ &= \frac{\pi^k r^{2k}}{k+1} \frac{1 - (1 - \varkappa^2)^{k+1}}{\varkappa^2}. \end{aligned}$$

Applying L'Hôpital's rule we get

$$\lim_{\varkappa \rightarrow 0} \frac{1 - (1 - \varkappa^2)^{k+1}}{\varkappa^2} = \lim_{\varkappa \rightarrow 0} \frac{(k+1)(1 - \varkappa^2)^k 2\varkappa}{2\varkappa} = (k+1) \lim_{\varkappa \rightarrow 0} (1 - \varkappa^2)^k = k+1,$$

hence

$$\mathbb{E}[A_0^k] = \pi^k r^{2k}.$$

Now we consider the case $1 < \varkappa < \infty$ ($r < R < \infty$). Here we have

$$\begin{aligned} \mathbb{E}[A_R^k] &= \frac{2}{R^2} \int_0^R A^k(\varrho) \varrho \, d\varrho = \frac{2}{R^2} \left(\int_0^r A^k(\varrho) \varrho \, d\varrho + \int_r^R A^k(\varrho) \varrho \, d\varrho \right) \\ &= \frac{2}{R^2} \left(\int_0^r \pi^k (r^2 - \varrho^2)^k \varrho \, d\varrho + \int_r^R \pi^k (\varrho^2 - r^2)^k \varrho \, d\varrho \right) \\ &= \frac{2\pi^k r^{2k+2}}{R^2} \left(\int_0^1 (1 - w^2)^k w \, dw + \int_1^{R/r} (w^2 - 1)^k w \, dw \right), \end{aligned}$$

hence

$$\begin{aligned} \mathbb{E}[A_{\varkappa}^k] &= \frac{2\pi^k r^{2k}}{\varkappa^2} \left(\frac{1}{2(k+1)} + \int_1^{\varkappa} (w^2 - 1)^k w \, dw \right) = \frac{2\pi^k r^{2k}}{\varkappa^2} \left(\frac{1}{2(k+1)} + \frac{(\varkappa^2 - 1)^{k+1}}{2(k+1)} \right) \\ &= \frac{\pi^k r^{2k}}{k+1} \frac{1 + (\varkappa^2 - 1)^{k+1}}{\varkappa^2}. \end{aligned} \quad \square$$

The graph of $\mathbb{E}[A_\varkappa]/r^2$ is shown in Fig. 3. For $\varkappa > 1$ we have

$$\frac{d}{d\varkappa} \mathbb{E}[A_\varkappa] = \frac{\pi r^2 (\varkappa^4 - 2)}{\varkappa^3}.$$

It follows that the expectation $\mathbb{E}[A_\varkappa]$ has its global minimum at point $\varkappa = \sqrt[4]{2} \approx 1.18921$ with value

$$\mathbb{E}[A_{\sqrt[4]{2}}] = (\sqrt{2} - 1)\pi r^2 \approx 1.30129 r^2.$$

Furthermore one finds that

$$\mathbb{E}[A_1] = \mathbb{E}[A_{\sqrt{2}}] = \frac{1}{2} \pi r^2 = \frac{1}{2} \tilde{A}(0) = \frac{1}{2} \tilde{A}(\sqrt{2}) = \frac{1}{2} \mathbb{E}[A_0].$$

For the variance of A_\varkappa , $\text{Var}[A_\varkappa] = \mathbb{E}[A_\varkappa^2] - \mathbb{E}[A_\varkappa]^2$, we get

$$\text{Var}[A_\varkappa] = \frac{\pi^2 r^4}{3} \frac{1 - (1 - \varkappa^2)^3}{\varkappa^2} - \frac{\pi^2 r^4}{4} \frac{[1 - (1 - \varkappa^2)^2]^2}{\varkappa^4} = \frac{\pi^2 r^4 \varkappa^4}{12}$$

if $0 \leq \varkappa \leq 1$, and

$$\text{Var}[A_\varkappa] = \frac{\pi^2 r^4}{3} \frac{1 + (\varkappa^2 - 1)^3}{\varkappa^2} - \frac{\pi^2 r^4}{4} \frac{[1 + (\varkappa^2 - 1)^2]^2}{\varkappa^4} = \pi^2 r^4 \left[-1 - \frac{1}{\varkappa^4} + \frac{2}{\varkappa^2} + \frac{\varkappa^4}{12} \right]$$

if $1 < \varkappa < \infty$. One finds that $\text{Var}[A_\varkappa]$ has local extrema at

$$\varkappa_1 \approx 1.06840 \quad \text{and} \quad \varkappa_2 \approx 1.30621$$

with values

$$\text{Var}[\varkappa_1] \approx 0.920036 \quad \text{and} \quad \text{Var}[\varkappa_2] \approx 0.703487,$$

respectively (see Fig. 4).

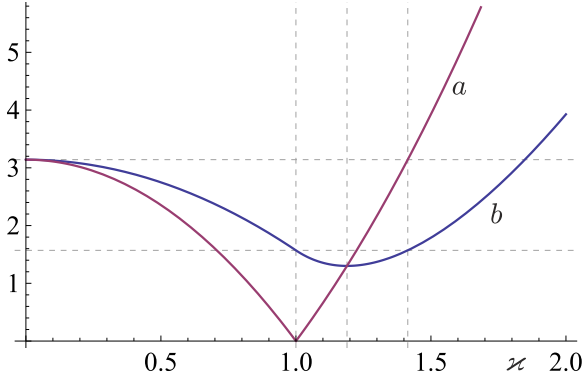


Fig. 3: a) $\tilde{A}(\varkappa)/r^2$ (see (2.3)), b) $\mathbb{E}[A_\varkappa]/r^2$

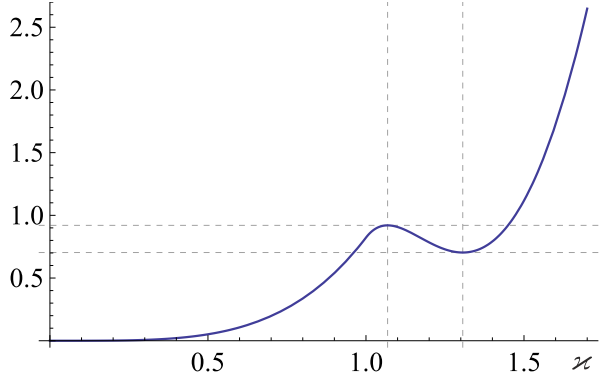


Fig. 4: $\text{Var}[A_\varkappa]/r^4$

3 Distribution of the area

Now we determine the distribution function

$$F_\varkappa(x) = \mathbb{P}[A_\varkappa \leq x]$$

of the random variable A_\varkappa . We have to distinguish the following three cases.

- 1) $0 \leq R \leq r$ ($0 \leq \varkappa \leq 1$): The smallest area of an ellipse is equal to $\pi(r^2 - R^2)$ and the biggest area equal to πr^2 . We have $A_R > x$ if P lies in an open disk with area $\pi \varrho^2$. From the first equation in (2.2), with $x = A$ we get

$$\varrho^2 = r^2 \left(1 - \frac{x}{\pi r^2}\right).$$

It follows that

$$\mathbb{P}[A_R > x] = \frac{\pi \varrho^2}{\pi R^2} = \frac{r^2}{R^2} \left(1 - \frac{x}{\pi r^2}\right)$$

and

$$\mathbb{P}[A_R \leq x] = 1 - \mathbb{P}[A_R > x] = 1 - \frac{r^2}{R^2} \left(1 - \frac{x}{\pi r^2}\right)$$

So we have

$$F_{\varkappa}(x) = \begin{cases} 0 & \text{if } -\infty < x < \pi r^2 (1 - \varkappa^2), \\ 1 - \frac{1}{\varkappa^2} \left(1 - \frac{x}{\pi r^2}\right) & \text{if } \pi r^2 (1 - \varkappa^2) \leq x < \pi r^2, \\ 1 & \text{if } \pi r^2 \leq x < \infty. \end{cases}$$

- 2) $r \leq R \leq \sqrt{2}r$ ($1 \leq \varkappa \leq \sqrt{2}$): For $0 \leq A_R \leq \pi(R^2 - r^2)$ we have $A_R > x$ if P lies

- a) in an open disk with area $\pi \varrho^2$ or
- b) in an open annulus with area $\pi(R^2 - \varrho'^2)$, where ϱ' is the ϱ in the second equation in (2.2).

It follows that

$$\begin{aligned} \mathbb{P}[A_R > x] &= \frac{\pi \varrho^2 + \pi(R^2 - \varrho'^2)}{\pi R^2} = 1 + \frac{\varrho^2 - \varrho'^2}{R^2} = 1 + \frac{r^2}{R^2} \left[1 - \frac{x}{\pi r^2} - \left(1 + \frac{x}{\pi r^2}\right)\right] \\ &= 1 - \frac{2x}{\pi R^2}, \end{aligned}$$

hence

$$\mathbb{P}[A_R \leq x] = 1 - \mathbb{P}[A_R > x] = \frac{2x}{\pi R^2}.$$

For $\pi(R^2 - r^2) \leq A_R \leq \pi r^2$ we have $A_R > x$ if P lies in an open disk with area $\pi \varrho^2$. Therefore, the distribution function is given by

$$F_{\varkappa}(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ \frac{2x}{\pi r^2 \varkappa^2} & \text{if } 0 \leq x < \pi r^2 (\varkappa^2 - 1), \\ 1 - \frac{1}{\varkappa^2} \left(1 - \frac{x}{\pi r^2}\right) & \text{if } \pi r^2 (\varkappa^2 - 1) \leq x < \pi r^2, \\ 1 & \text{if } \pi r^2 \leq x < \infty. \end{cases}$$

- 3) $\sqrt{2}r \leq R < \infty$ ($\sqrt{2} \leq \varkappa < \infty$): One easily finds

$$F_{\varkappa}(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ \frac{2x}{\pi r^2 \varkappa^2} & \text{if } 0 \leq x < \pi r^2, \\ \frac{1}{\varkappa^2} \left(1 + \frac{x}{\pi r^2}\right) & \text{if } \pi r^2 \leq x < \pi r^2 (\varkappa^2 - 1), \\ 1 & \text{if } \pi r^2 (\varkappa^2 - 1) \leq x < \infty. \end{cases}$$

Now we determine the moments of the random variabe A_{\varkappa} in an alternative way:

- 1) $0 \leq \varkappa \leq 1$: The density function is given by $f_{\varkappa}(x) = 1/(\pi r^2 \varkappa^2)$ if $\pi r^2 (1 - \varkappa^2) \leq x < \pi r^2$, and $f_{\varkappa}(x) = 0$ if the area is outside this interval. So we have

$$\mathbb{E}[A_{\varkappa}^k] = \int_{\pi r^2(1-\varkappa^2)}^{\pi r^2} x^k f_{\varkappa}(x) dx = \frac{1}{\pi r^2 \varkappa^2} \int_{\pi r^2(1-\varkappa^2)}^{\pi r^2} x^k dx = \frac{\pi^k r^{2k}}{k+1} \frac{1 - (1 - \varkappa^2)^{k+1}}{\varkappa^2}.$$

- 2) $1 \leq \varkappa \leq \sqrt{2}$: The restriction of the density function $f_{\varkappa}(x)$ to the interval $0 \leq x \leq \pi r^2$ is given by

$$f_{\varkappa}(x) = \frac{2}{\pi r^2 \varkappa^2} \text{ if } 0 \leq x < \pi r^2(\varkappa^2 - 1), \quad \text{and} \quad f_{\varkappa}(x) = \frac{1}{\pi r^2 \varkappa^2} \text{ if } \pi r^2(\varkappa^2 - 1) \leq x \leq \pi r^2.$$

It follows that

$$\begin{aligned} \mathbb{E}[A_{\varkappa}^k] &= \frac{2}{\pi r^2 \varkappa^2} \int_0^{\pi r^2(\varkappa^2-1)} x^k dx + \frac{1}{\pi r^2 \varkappa^2} \int_{\pi r^2(\varkappa^2-1)}^{\pi r^2} x^k dx \\ &= \frac{2\pi^k r^{2k} (\varkappa^2 - 1)^{k+1}}{(k+1)\varkappa^2} + \frac{\pi^k r^{2k} [1 - (\varkappa^2 - 1)^{k+1}]}{(k+1)\varkappa^2} = \frac{\pi^k r^{2k}}{k+1} \frac{1 + (\varkappa^2 - 1)^{k+1}}{\varkappa^2}. \end{aligned}$$

- 3) $\sqrt{2} \leq \varkappa < \infty$: One finds

$$\begin{aligned} \mathbb{E}[A_{\varkappa}^k] &= \frac{2}{\pi r^2 \varkappa^2} \int_0^{\pi r^2} x^k dx + \frac{1}{\pi r^2 \varkappa^2} \int_{\pi r^2}^{\pi r^2(\varkappa^2-1)} x^k dx \\ &= \frac{2\pi^k r^{2k}}{(k+1)\varkappa^2} + \frac{\pi^k r^{2k} [(\varkappa^2 - 1)^{k+1} - 1]}{(k+1)\varkappa^2} = \frac{\pi^k r^{2k}}{k+1} \frac{1 + (\varkappa^2 - 1)^{k+1}}{\varkappa^2}. \end{aligned}$$

4 Moments of the perimeter

By analogy to the determination of $\mathbb{E}[A_R^k]$, we get the moments of the perimeter with

$$\begin{aligned} \mathbb{E}[U_R^k] &= \left(\int_{P \in D_R} u^k dP \right) / \left(\int_{P \in D_R} dP \right) = \frac{1}{\pi R^2} \int_{\alpha=0}^{2\pi} \int_{\varrho=0}^R u^k(\varrho) \varrho d\varrho d\alpha \\ &= \frac{2}{R^2} \int_{\varrho=0}^R u^k(\varrho) \varrho d\varrho, \end{aligned}$$

where $u(\varrho)$ is the perimeter of an ellipse generated by a point $P \in D_R$ with distance ϱ from Ω . The length of the semi-major axis is given by $r + \varrho$, and the length of the semi-minor axis by $r - \varrho$ or $\varrho - r$. In both cases we have

$$u(\varrho) = 4(r + \varrho) E \left(\frac{2\sqrt{r\varrho}}{r + \varrho} \right), \quad (4.1)$$

where

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

is the complete elliptic integral of the second kind with modulus k , $0 \leq k \leq 1$. So we have

$$\mathbb{E}[U_R^k] = \frac{2^{2k+1}}{R^2} \int_0^R \varrho (r + \varrho)^k E^k \left(\frac{2\sqrt{r\varrho}}{r + \varrho} \right) d\varrho. \quad (4.2)$$

By substituting $w = \varrho/r$, the perimeter (4.1) may be written as

$$\tilde{u}(w) = rh(w), \quad \text{where} \quad h(w) := 4(w+1) E\left(\frac{2\sqrt{w}}{w+1}\right), \quad (4.3)$$

and, with $U_{\varkappa} \equiv U_R$, $\varkappa = R/r$, the integral (4.2) becomes

$$\mathbb{E}[U_{\varkappa}^k] = \frac{2^{2k+1} r^k}{\varkappa^2} \int_0^{\varkappa} w (w+1)^k E^k\left(\frac{2\sqrt{w}}{w+1}\right) dw. \quad (4.4)$$

Theorem 4.1. *The expectation of the random perimeter $U_{\varkappa} = U_R$, $\varkappa = R/r$, of an ellipse generated by a random point $P \in D_R$ (P uniformly distributed over the area of the disk D_R) is given by*

$$\mathbb{E}[U_{\varkappa}] = \begin{cases} 2\pi r & \text{if } \varkappa = 0, \\ \frac{8r}{9\varkappa^2} [(7\varkappa^2 + 1) E(\varkappa) + (3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa)] & \text{if } 0 < \varkappa < 1, \\ 64r/9 & \text{if } \varkappa = 1, \\ \frac{8r}{9\varkappa} [(7\varkappa^2 + 1) E(\varkappa^{-1}) - 4(\varkappa^2 - 1) K(\varkappa^{-1})] & \text{if } \varkappa > 1 \end{cases}$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

is the complete elliptic integral of the first kind with modulus k , $0 \leq k \leq 1$. The expectations for the cases $0 < \varkappa < 1$ and $\varkappa > 1$ can be subsumed in the formula

$$\mathbb{E}[U_{\varkappa}] = \frac{4(\varkappa + 1)r}{9\varkappa^2} \left[(7\varkappa^2 + 1) E\left(\frac{2\sqrt{\varkappa}}{\varkappa + 1}\right) - (\varkappa - 1)^2 K\left(\frac{2\sqrt{\varkappa}}{\varkappa + 1}\right) \right].$$

Proof. First, we consider the case $0 < \varkappa < 1$. Using the relation

$$E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k} [2E(k) - (1-k^2) K(k)] \quad (4.5)$$

([3, Vol. 2, p. 299], Eq. (8.126.4)), (4.4) becomes

$$\mathbb{E}[U_{\varkappa}] = \frac{8r}{\varkappa^2} \left[2 \int_0^{\varkappa} w E(w) dw - \int_0^{\varkappa} w K(w) dw + \int_0^{\varkappa} w^3 K(w) dw \right].$$

With

$$\begin{aligned} \int E(k) k dk &= \frac{1}{3} [(1+k^2) E(k) - (1-k^2) K(k)], \\ \int K(k) k dk &= E(k) - (1-k^2) K(k), \\ \int K(k) k^3 dk &= \frac{1}{9} [(4+k^2) E(k) - (1-k^2) (4+3k^2) K(k)] \end{aligned}$$

(Equations (5.112.4), (5.112.3), (5.112.5) in [3, Vol. 2, p. 13]) we get

$$\mathbb{E}[U_{\varkappa}] = \frac{8r}{9\varkappa^2} [(7\varkappa^2 + 1) E(\varkappa) + (3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa)], \quad 0 < \varkappa < 1. \quad (4.6)$$

For $\varkappa = 0$ we have $E(\varkappa) = \pi/2 = K(\varkappa)$, hence

$$(7\varkappa^2 + 1) E(\varkappa) + (3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa) = 0.$$

So $\mathbb{E}[U_\varkappa]$ has the indeterminate form $0/0$. Taking

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k} \quad \text{and} \quad \frac{dK(k)}{dk} = \frac{E(k) - (1 - k^2)K(k)}{k(1 - k^2)}$$

(Equations (8.123.4), (8.123.2) in [3, Vol. 2, p. 298]) into account, applying L'Hôpital's rule twice gives

$$\lim_{\varkappa \rightarrow 0} \mathbb{E}[U_\varkappa] = 4r \lim_{\varkappa \rightarrow 0} [3E(\varkappa) + 2(\varkappa^2 - 1)K(\varkappa)] = 4r \left[3 \frac{\pi}{2} - 2 \frac{\pi}{2} \right] = 2\pi r.$$

Now we consider the limit of (4.6) for $\varkappa \rightarrow 1$ (and hence $R \rightarrow r$). We have $E(1) = 1$, $K(1) = \infty$. For $\varkappa = 1$,

$$(3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa)$$

has the indeterminate form $0 \cdot \infty$. *Mathematica* finds

$$\lim_{\varkappa \rightarrow 1-} (3\varkappa^4 - 2\varkappa^2 - 1)K(\varkappa) = 0.$$

It follows that

$$\lim_{\varkappa \rightarrow 1-} \mathbb{E}[U_\varkappa] = \frac{8r}{9 \cdot 1} [(7 + 1) + 0] = \frac{64r}{9}.$$

Now we consider the case $\varkappa = R/r > 1$. Here we have

$$\begin{aligned} \mathbb{E}[U_\varkappa] &= \frac{64r}{9\varkappa^2} + \frac{8}{R^2} \int_{\varrho=r}^R \varrho (r + \varrho) E\left(\frac{2\sqrt{r\varrho}}{r + \varrho}\right) d\varrho \\ &= \frac{64r}{9\varkappa^2} + \frac{8}{R^2} \int_{\varrho=r}^R \varrho^2 \left(\frac{r}{\varrho} + 1\right) E\left(\frac{2\sqrt{r/\varrho}}{(r/\varrho) + 1}\right) d\varrho. \end{aligned}$$

The substitution

$$v = \frac{r}{\varrho}, \quad dv = -\frac{r}{\varrho^2} d\varrho, \quad d\varrho = -\frac{r}{v^2} dv, \quad \varrho = r \implies v = 1, \quad \varrho = R \implies v = r/R = 1/\varkappa$$

gives

$$\mathbb{E}[U_\varkappa] = \frac{64r}{9\varkappa^2} + \frac{8r}{\varkappa^2} \int_{1/\varkappa}^1 \frac{v + 1}{v^4} E\left(\frac{2\sqrt{v}}{1 + v}\right) dv.$$

Applying (4.5), we get

$$\mathbb{E}[U_\varkappa] = \frac{64r}{9\varkappa^2} + \frac{8r}{\varkappa^2} \left[2 \int_{1/\varkappa}^1 \frac{E(v)}{v^4} dv - \int_{1/\varkappa}^1 \frac{K(v)}{v^4} dv + \int_{1/\varkappa}^1 \frac{K(v)}{v^2} dv \right]$$

From [3, Vol. 2, pp. 13-14], Equations (5.112.12), (5.112.9), we know that

$$\begin{aligned} \int \frac{E(k)}{k^4} dk &= \frac{1}{9k^3} [2(k^2 - 2)E(k) + (1 - k^2)K(k)], \\ \int \frac{K(k)}{k^2} dk &= -\frac{E(k)}{k}. \end{aligned}$$

Mathematica finds

$$\int_{1/\varkappa}^1 \frac{K(v)}{v^4} dv = \frac{1}{9} [-5 + \varkappa(\varkappa^2 + 4)E(\varkappa^{-1}) + 2\varkappa(\varkappa^2 - 1)K(\varkappa^{-1})].$$

So we get

$$\mathbb{E}[U_{\varkappa}] = \frac{8r}{9\varkappa} [(7\varkappa^2 + 1) E(\varkappa^{-1}) - 4(\varkappa^2 - 1) K(\varkappa^{-1})], \quad \varkappa > 1.$$

For $\varkappa = 1$,

$$(\varkappa^2 - 1) K(\varkappa^{-1})$$

has the indeterminate form $0 \cdot \infty$. *Mathematica* finds

$$\lim_{\varkappa \rightarrow 1+} (\varkappa^2 - 1) K(\varkappa^{-1}) = 0.$$

It follows that

$$\lim_{\varkappa \rightarrow 1+} \mathbb{E}[U_{\varkappa}] = \frac{8r}{9 \cdot 1} [(7 + 1) - 0] = \frac{64r}{9}.$$

Using the relations (4.5) and

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)K(k)$$

[3, Vol. 2, p. 299], Eq. (8.126.3), for $0 < \varkappa < 1$ we get

$$\begin{aligned} f(\varkappa) &:= \frac{4(\varkappa+1)r}{9\varkappa^2} \left[(7\varkappa^2 + 1) E\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right) - (\varkappa-1)^2 K\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right) \right] \\ &= \frac{8r}{9\varkappa^2} [(7\varkappa^2 + 1) E(\varkappa) + (3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa)] = \mathbb{E}[U_{\varkappa}]. \end{aligned}$$

For $\varkappa > 1$ we have

$$\begin{aligned} E\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right) &= E\left(\frac{2\sqrt{\varkappa^{-1}}}{1+\varkappa^{-1}}\right) = \frac{1}{1+\varkappa^{-1}} [2E(\varkappa^{-1}) - (1-\varkappa^{-2}) K(\varkappa^{-1})] \\ &= \frac{2\varkappa}{\varkappa+1} E(\varkappa^{-1}) - \frac{\varkappa-1}{\varkappa} K(\varkappa^{-1}) \end{aligned}$$

and

$$K\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right) = K\left(\frac{2\sqrt{\varkappa^{-1}}}{1+\varkappa^{-1}}\right) = \frac{\varkappa+1}{\varkappa} K(\varkappa^{-1}).$$

It follows that

$$\begin{aligned} f(\varkappa) &= \frac{4r}{9\varkappa^2} [2\varkappa(7\varkappa^2 + 1) E(\varkappa^{-1}) - 8\varkappa(\varkappa^2 - 1) K(\varkappa^{-1})] \\ &= \frac{8r}{9\varkappa} [(7\varkappa^2 + 1) E(\varkappa^{-1}) - 4(\varkappa^2 - 1) K(\varkappa^{-1})] = \mathbb{E}[U_{\varkappa}]. \end{aligned}$$

For $\varkappa = 0$,

$$(7\varkappa^2 + 1) E\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right) - (\varkappa-1)^2 K\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right) = 0,$$

hence $f(0)$ has the indeterminate form $0/0$. For $\varkappa = 1$,

$$(\varkappa-1)^2 K\left(\frac{2\sqrt{\varkappa}}{\varkappa+1}\right)$$

has the indeterminate form $0 \cdot \infty$. *Mathematica* finds

$$\lim_{\varkappa \rightarrow 0} f(\varkappa) = 2\pi r \quad \text{and} \quad \lim_{\varkappa \rightarrow 1} f(\varkappa) = \frac{64r}{9}.$$

□

The graph of $\mathbb{E}[U_{\varkappa}]/r$ is shown in Fig. 5. *Mathematica* finds the series expansion

$$\mathbb{E}[U_{\varkappa}] = \pi r \left(\frac{4\varkappa}{3} + \frac{1}{\varkappa} - \frac{1}{16\varkappa^3} \right) + O(\varkappa^{-5})$$

about the point $\varkappa = \infty$. For abbreviation we put

$$s(\varkappa) := \pi r \left(\frac{4\varkappa}{3} + \frac{1}{\varkappa} - \frac{1}{16\varkappa^3} \right). \quad (4.7)$$

$s(\varkappa)$ provides a very good approximation for $\mathbb{E}[U_{\varkappa}]$ even for relatively small values of \varkappa (see Fig. 5). One finds that

$$\begin{aligned} \mathbb{E}[U_1] - s(1) &\approx -2.29222 \cdot 10^{-2} r, & \mathbb{E}[U_2] - s(2) &\approx -5.44238 \cdot 10^{-4} r, \\ \mathbb{E}[U_{10}] - s(10) &\approx -1.64009 \cdot 10^{-7} r. \end{aligned}$$

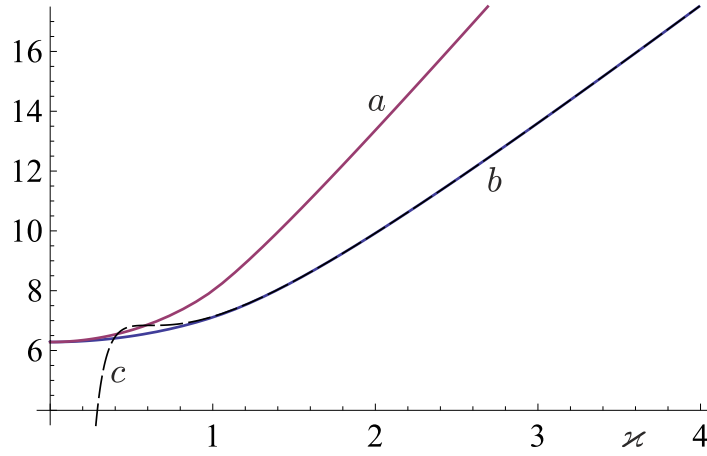


Fig. 5: a) $\tilde{u}(\varkappa)/r$ (see (4.3)), b) $\mathbb{E}[U_{\varkappa}]/r$, c) $s(\varkappa)/r$

5 Distribution of the perimeter

With (4.3) the maximum perimeter in the disk of radius R is given by

$$\tilde{u}(\varkappa) = rh(\varkappa) \quad \text{with} \quad \varkappa = R/r.$$

Let $G_{\varkappa}(u) = \mathbb{P}[U_{\varkappa} \leq u]$ be the distribution function of the random variable U_{\varkappa} . One easily finds from geometrical considerations

$$G_{\varkappa}(u) = \begin{cases} 0 & \text{if } -\infty < u < 2\pi r, \\ \frac{w^2(u)}{\varkappa^2} & \text{if } 2\pi r \leq u < rh(\varkappa), \\ 1 & \text{if } rh(\varkappa) \leq u < \infty, \end{cases} \quad (5.1)$$

where $w = w(u)$ is the solution of

$$rh(w) = u.$$

The Figures 6-11 show examples for graphs of distribution functions G_{\varkappa} and corresponding density functions g_{\varkappa} (multiplied with r). The density functions are obtained by numerical differentiation of the distribution functions. For comparison the distribution function and the density function (multiplied with r) of the uniform distribution with support interval $2\pi r \leq u \leq rh(\varkappa)$ are shown (dashed lines).

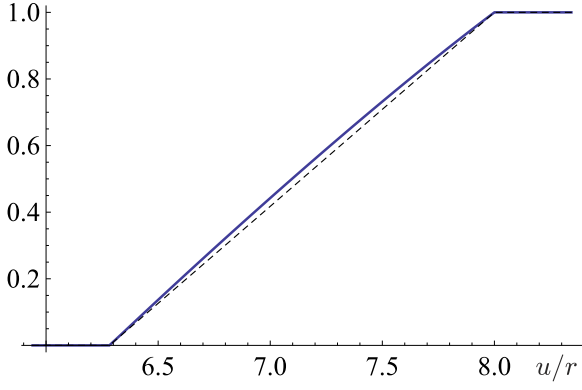


Fig. 6: Distribution function $G_1(u)$

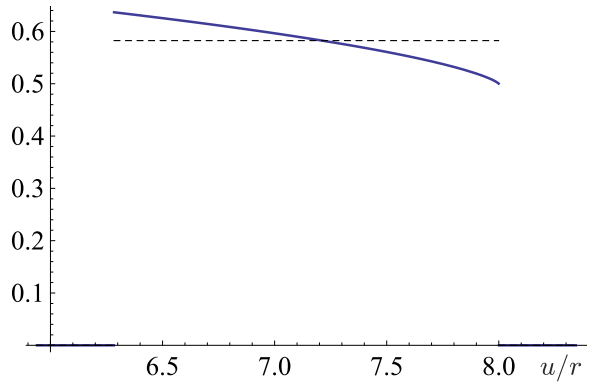


Fig. 7: Density function $rg_1(u)$

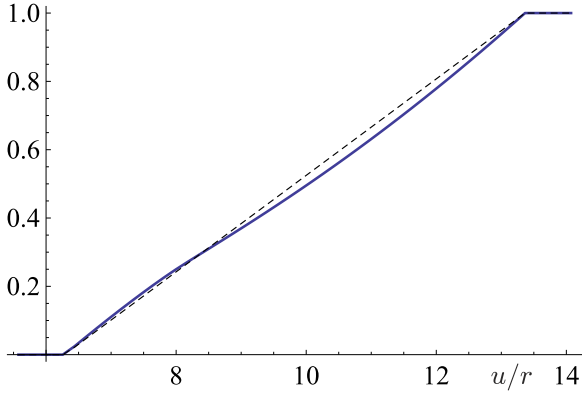


Fig. 8: Distribution function $G_2(u)$

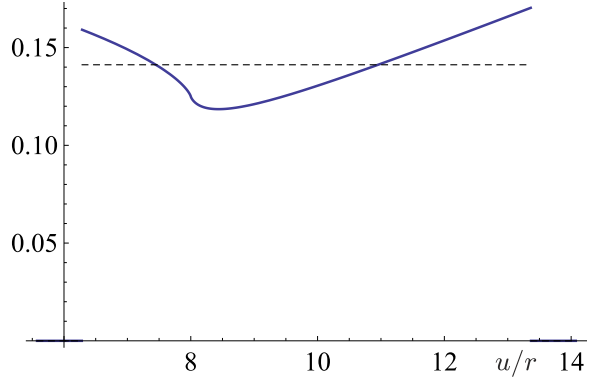


Fig. 9: Density function $rg_2(u)$

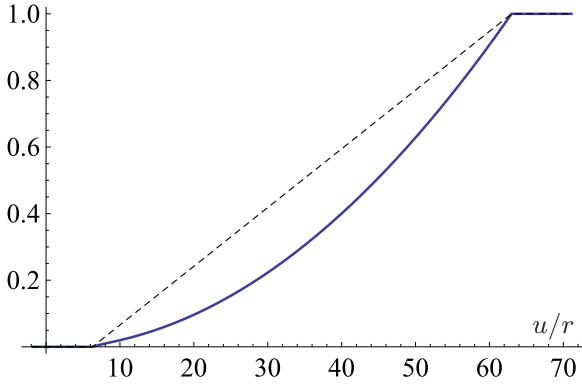


Fig. 10: Distribution function $G_{10}(u)$

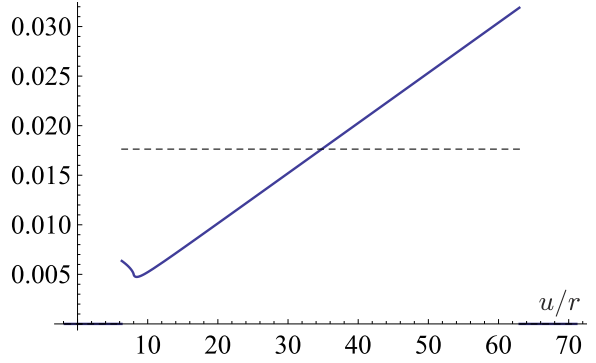


Fig. 11: Density function $rg_{10}(u)$

For the k -th moment of the perimeter U_{\varkappa} we have the Stieltjes integral

$$\mathbb{E}[U_{\varkappa}^k] = \int_{-\infty}^{\infty} u^k dG_{\varkappa}(u) = \int_{2\pi r}^{rh(\varkappa)} u^k dG_{\varkappa}(u).$$

Integration by parts yields

$$\mathbb{E}[U_{\varkappa}^k] = u^k G_{\varkappa}(u) \Big|_{2\pi r}^{rh(\varkappa)} - k \int_{2\pi r}^{rh(\varkappa)} u^{k-1} G_{\varkappa}(u) du = r^k h^k(\varkappa) - k \int_{2\pi r}^{rh(\varkappa)} u^{k-1} G_{\varkappa}(u) du.$$

From (5.1) it follows that

$$\mathbb{E}[U_{\varkappa}^k] = r^k h^k(\varkappa) - \frac{k}{\varkappa^2} \int_{2\pi r}^{rh(\varkappa)} u^{k-1} w^2(u) du.$$

The substitution $u = rx$ gives

$$\mathbb{E}[U_{\varkappa}^k] = r^k \left[h^k(\varkappa) - \frac{k}{\varkappa^2} \int_{2\pi}^{h(\varkappa)} x^{k-1} \tilde{w}^2(x) dx \right], \quad (5.2)$$

where $\tilde{w} = \tilde{w}(x)$ is the solution of the equation

$$rh(\tilde{w}) = rx \implies h(\tilde{w}) = x$$

for given value of $x \in [2\pi, h(\varkappa)]$.

Table 1 shows examples for numerical values of $\mathbb{E}[U_{\varkappa}^k]/r^k$ which are obtained by numerical integration of (4.4) and (5.2) using *Mathematica*. The values for $k = 1$ also directly follow from Theorem 4.1.

k	$\mathbb{E}[U_2^k]/r^k$	$\mathbb{E}[U_3^k]/r^k$
1	9.9232888058187711084	13.606226799878091189
2	102.96648551991466206	199.63369601685873413
3	1110.1715673108248830	3094.8106779481943393
4	12355.455260295394611	49903.060116964320575
5	141074.96324382298144	827860.92690590516817
6	$1.6440752317143806830 \cdot 10^6$	$1.4025517639333515857 \cdot 10^7$
7	$1.9474969898992456378 \cdot 10^7$	$2.4146079537555357859 \cdot 10^8$
8	$2.3373182556510259688 \cdot 10^8$	$4.2096527665994963840 \cdot 10^9$
9	$2.8351332586002858086 \cdot 10^9$	$7.4140462419039734138 \cdot 10^{10}$
10	$3.4691601000674476672 \cdot 10^{10}$	$1.3167229871972133743 \cdot 10^{12}$

Table 1: Numerical values of $\mathbb{E}[U_{\varkappa}^k]/r^k$ for $\varkappa = 2, 3$

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